# Time and systems

Robert Vallée

*World Organisation of Systems and Cybernetics, Earley, Reading, UK*

**Abstract**

**Purpose** – To give a mathematical expression of what could be called the internal time of a dynamical system, a time which is different from the external or reference time.

**Design/methodology/approach** – The paper introduces a general mathematical definition of internal duration and so of internal time. Then we consider the case of an explosion followed by an implosion, which we apply to cosmology and physiology. The case of diffusion is also presented.

**Findings** – The internal time is generally different from the reference time. In certain cases to a finite reference duration may correspond an infinite internal duration.

**Research limitations/implications** – Our formulations may help to understand certain aspects of cosmology, physiology and more generally of the evolution of dynamical systems.

**Practical implications** – For example, the physiology of ageing.

**Originality/value** – The consideration of the square of the speed of evolution, at instant $t$, of a dynamical system for measuring the internal duration of interval $(t, t + \mathrm{d}t)$ is original, as well as its consequences.

**Keywords** Cosmology, Thermal diffusion, Dynamics, Time measurement

**Paper type** Research paper

## 1. Introduction

Our purpose is to propose a definition of the *internal time* or intrinsic time (Vallée, 1981, 1986, 1991), of a dynamical system evolving independently of any environment. The notion of internal time $\theta$ is opposed to that of external time, or *reference time* $t$, taken for granted and which is used in the evolution equation. The basic idea is that the internal time does not elapse if the state of the system does not vary, a conception close to that of Aristotle for whom time ceases to be known when the "soul" does not change.

So if $y(t)$, belonging to $\mathrm{R}^\mathrm{N}$ (or $\mathrm{C}^\mathrm{N}$), is the state of the system at reference instant $t$, any positive and increasing function, null for 0, of a norm of $y(t)/\mathrm{d}t$, is a measure of the state of movement of the system at instant $t$. We make the most simple choice, that of the square of the Euclidian norm, represented by $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$. So we define the *internal duration* $\mathrm{d}(t_1,t_2)$ of interval $(t_1, t_2)$, of *reference duration* $t_2 - t_1$, by (Vallée, 1996, 2001).

$$\mathrm{d}(t_1, t_2) = \int_{t_1,t_2} \|\mathrm{d}y(s)/\mathrm{d}s\|^2 \,\mathrm{d}s \tag{1}$$

So if $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$ is equal to 0 on the interval, the internal duration is 0, and if $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$ is equal to 1, the internal duration is equal to the reference duration. In short, the higher the values of $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$ on the interval, the longer the internal duration. So $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$ represents the *weight of reference instant t*.

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We can now define the *internal time* $\theta(t)$ by

$$\theta(t) = \mathrm{d}(t_0, t) = \int_{t_0,t} \|\mathrm{d}y(s)/\mathrm{d}s\|^2 \,\mathrm{d}s, \tag{2}$$

where $t_0$ is any reference instant of the life of the system. Of course we have

$$\mathrm{d}(t_1, t_2) = \theta(t_2) - \theta(t_1), \quad t_1 < t_2 \tag{3}$$

## 2. Explosions and implosions

We call *explosion* the evolution of a system whose modulus of the state vector starts with value 0 at $t = 0$ then increases with $t$, and such that the modulus of its speed vector starts with value $+\infty$ at $t = 0$. The first instants of the evolution of the system have an exceptional importance since the weight $\|\mathrm{d}y(t)/\mathrm{d}t\|^2$ of instant $t$ tends to $+\infty$ when $t$ tends to $0+$. We have here an idealisation as well as in the case of what we call *implosion* where $\|y(t)\|$ decreases with $t$ and attains value 0 at the final instant while $\|\mathrm{d}y(t)/\mathrm{d}t\|$ tends to $+\infty$. A system may be explosive at the beginning and implosive at the end, in that case we say that we have an *explosion-implosion*. For the sake of simplicity we shall suppose now that $y(t)$ is a mere scalar.

We shall start with an *explosion-implosion* (Vallée, 1996, 2001) defined by the differential equation

$$\mathrm{d}y(t)/\mathrm{d}t = q/p\,\mathrm{sgn}(p - t)(q^2 - y^2(t))^{1/2}/y(t), \quad y(0) = 0, \quad p \text{ and } q > 0, \quad t\,\varepsilon\,[0, 2p], \tag{4}$$

where $\mathrm{sgn}(p - t)$ is the sign of $p - t$. The solution of this equation is given by function

$$y(t) = q/p\,(p^2 - (p - t)^2)^{1/2} \tag{5}$$

whose graph is an half-ellipse of great axis $2p$ and small axis $2q$. We shall say that we have an *elliptic explosion-implosion*. When $t$ varies from 0 to $2p$, $y(t)$ increases from 0 to $q$ then decreases from $q$ to 0, with a speed of infinite absolute value at 0 and $2p$. The square of the speed of evolution is given by

$$(\mathrm{d}y(t)/\mathrm{d}t)^2 = q^2/p^2\,(p - t)^2/t(2p - t) = q^2/2p\,(1/t - 2/p + 1/2p - t)$$

which shows that the *weight of instant* $t$ is infinite at the beginning ($t = 0$) and at the end ($t = 2p$) of the life of the system. If we integrate $(\mathrm{d}y(t)/\mathrm{d}t)^2$ from $t_1$ to $t_2$ we obtain the *internal duration* of the reference time interval $(t_1, t_2)$

$$\mathrm{d}(t_1, t_2) = q^2/2p\,(\mathrm{Log}(t_2/t_1) - 2(t_2 - t_1)/p - \mathrm{Log}(2p - t_2/2p - t_1)), \tag{6}$$

and, remembering that an associated *internal time* $\theta(t)$ is defined up to an additive constant, we can choose, for the sake of simplification,

$$\theta(t) = \mathrm{d}(p, t) - q^2/p = q^2/2p\,(\mathrm{Log}\,t - 2t/p - \mathrm{Log}(2p - t)). \tag{7}$$

We see that when the *reference time* $t$ varies from 0 to $2p$, generating a finite *reference duration* equal to $2p$, the internal time $\theta$ varies from $-\infty$ to $+\infty$ generating an infinite *internal duration*. The initial instant 0 is pushed back to $-\infty$ and the final instant $2p$ is pushed forward to $+\infty$. The internal duration of any interval $(0,t)$ is infinite as well as the internal duration of any interval $(t, 2p)$.

We consider now the differential equation

$$\mathrm{d}y(t)/\mathrm{d}t = q/p\,(q^2 + y^2(t))^{1/2}/y(t), \quad y(0) = 0, \quad p \text{ and } q > 0, \quad t\,\varepsilon\,[0, +\infty]. \tag{8}$$

Its solution is given by the function

$$y(t) = q/p\,((p+t)^2 - p^2)^{1/2} \tag{9}$$

whose graph is the right part of an half-hyperbola of great axis $2p$ and "small axis" $2q$. We shall say that we have an *hyperbolic explosion*: $y(t)$ increases from 0 to $+\infty$ with an infinite speed at instant 0 and, for the great values of $t$, $y(t)$ behaves like $q/p\,(t+p)$. The square of the speed is given by

$$(\mathrm{d}y(t)/\mathrm{d}t)^2 = q^2/p^2\,(p+t)^2/t(t+2p) = q^2/2p\,(1/t + 2/p - 1/2p + t).$$

Calculations, similar to those of the elliptic case, give the internal duration of interval $(t_1, t_2)$ and consequently, an *internal time*

$$\theta(t) = q^2/2p\ (\mathrm{Log}\,t + 2t/p - \mathrm{Log}(2p+t)). \tag{10}$$

When $t$ varies from 0 to $+\infty$, $\theta(t)$ varies from $-\infty$ to $+\infty$, the initial instant 0 being pushed back to $-\infty$. Any interval $(0, t)$, of reference duration $t$, has an infinite internal duration. Moreover $\theta(t)$ behaves as $q^2/2p\ \mathrm{Log}\,t$ for $t$ small and as $q^2/p^2\ t$ for great values of $t$.

An intermediary case, which we call *parabolic explosion* (Vallée, 1996, 2001), is obtained when $p$ and $q$ tend to infinity while $q^2/p$ keeps a constant value $2h$. Starting indifferently from equation (4) or (8), we obtain

$$\mathrm{d}y(t)/\mathrm{d}t = 2h/y(t), \quad y(0) = 0, \quad h > 0. \tag{11}$$

The solution is given by function

$$y(t) = 2(ht)^{1/2}, \tag{12}$$

whose graph is a half-parabola of parameter $h$. We see that $y(t)$ increases from 0 to $+\infty$ with an infinite initial speed. Then we have

$$(\mathrm{d}y(t)/\mathrm{d}t)^2 = h/t,$$

the *weight of instant t* is infinite at the initial instant and tends to 0 when $t$ tends to $+\infty$. The calculation of *internal duration* generates an *internal time*

$$\theta(t) = h\,\mathrm{Log}\,t, \tag{13}$$

the initial instant 0 is pushed back to $-\infty$ and any interval $(0,t)$ has an infinite *internal duration* $\theta$.

### 3. Physiology

We can interpret an *elliptic explosion-implosion* as the evolution of a living being whose birth may be compared to a kind of explosion and the end of life as an involution, or a kind of implosion, more or less quick. In our model the implosive part is symmetrical with the explosive one. This is not very realistic, since it seems that the implosive part must be shorter. Nevertheless if we consider only the qualitative aspect of the conclusions, we can conclude that, from the internal time point of view, the initial instant (conception) is pushed back to $-\infty$ and the final instant (death) is pushed forward to $+\infty$ (Vallée, 1991). The first part of this qualitative conclusion is in accordance with the natural feeling of a human being (if we consider this case) of not having any beginning. The second part is more controversial, nevertheless it has been more or less considered, with approaches different from ours, by some authors (Lévy, 1969; Lévi, 1975).

We can also consider the case of a *parabolic explosion* limited to the instant of death. The initial instant is pushed back to $-\infty$ and the *internal time* is proportional to the logarithm of the elapsed reference time. It elapses slowly at the beginning and more and more quickly near the end. This is close to the ideas of Lecomte du Noüy (1936). For him the physiological duration of an usual time interval, of given length, is proportional to the speed of healing of wounds. This speed varying roughly as the inverse of age, a logarithmic physiological time is generated.

Here a remark seems necessary in order to avoid apparent paradoxes: *the internal time* of a conscious being may be different from the *perceived internal time*.

### 4. Cosmology

We shall now interpret the notion of *internal duration* in cosmology. The cases of elliptic explosion-implosion, parabolic or hyperbolic explosion have common traits with cosmological models with primordial explosion followed by final implosion or with primordial explosion only.

Generally speaking, the differential equation giving the evolution of the universe, whose state at instant $t$ is described by the cosmological scale factor $R(t)$, representing in certain cases the radius of the universe, is, according to Lemaître, Friedmann, Robertson (Berry, 1989).

$$(\mathrm{d}R(t)/\mathrm{d}t)^2 = 8\pi G/3\,\rho(t)R^2(t) - kc^2 + \Lambda/3\ R^2(t), \quad R(0) = 0, \tag{14}$$

where $G$ is the gravitational constant, $c$ the speed of light, $k$ the index of curvature ($k = -1$, space with negative curvature; $k = 0$, flat space; $k = +1$, positive curvature, then $R(t)$ may be considered as the radius of the universe), $\Lambda$ the cosmic constant or cosmic repulsion term, $\rho(t)$ the density of matter or its material equivalent in case of pure radiation. In the material case $\rho(t) = a/R^3(t)$ and in the case of pure radiation it is equal to $b/R^4(t)$, $a$ and $b$ being constants.

Equation (4), corresponding to an *elliptic explosion-implosion*, gives if we consider the square of the two members

$$(\mathrm{d}y(t)/\mathrm{d}t)^2 = q^4/p^2/y^2(t) - q^2/p^2.$$

If we substitute $R(t)$ to $y(t)$, the above equation takes one of the possible forms of equation (14) if $\rho(t) = b/R^4(t)$, $k = +1$, $\Lambda = 0$. We have a case of pure radiation with positive curvature and null cosmical constant. More precisely $q = cp$ and $p = 1/c^2\ (8\pi G/3\,b)^{1/2}$. Then

$$(\mathrm{d}R(t)/\mathrm{d}t)^2 = 8\pi G/3\,b/R^2(t) - c^2. \tag{15}$$

The *internal time* of this system, which we propose to call *generalized cosmological time* (Vallée, 1995, 1996, 2001) is then given, according to equation (7), by

$$\theta(t) = c^2 p/2\,(\mathrm{Log}\,t - 2t/p + \mathrm{Log}(2p - t)). \tag{16}$$

The initial reference instant $t = 0$ (big bang) is pushed back to $-\infty$ and the final reference instant $t = 2p$ (big crunch) is pushed forward to $+\infty$.

But classically a cosmological model with pure radiation is accepted mainly as an approximation valid when the density of matter ($a/R^3(t)$) is negligible compared to the (equivalent) density of matter of pure radiation ($b/R^4(t)$). This happens when $R(t)$ is small so when $t$ is close to 0. In that case $-kc^2$ is negligible as well as $\Lambda/3R^2$ and so it is not even necessary to suppose that $\Lambda = 0$. We then have

$$(\mathrm{d}R(t)/\mathrm{d}t)^2 = 8\pi G/3\,b/R^2(t). \tag{17}$$

This equation is that of the radiation-dominated era, at the beginning of the universe, or that of an evolution with pure radiation, null cosmic constant and flat universe. It corresponds to a *parabolic explosion* whose differential equation, after taking the square of its two members, gives

$$(\mathrm{d}y(t)/\mathrm{d}t)^2 = 4h^2/y^2(t).$$

We have just to substitute $R$ to $y$ and $(2\pi G/3b)^{1/2}$ to $h$. According to equation (13), the *internal time* of this universe is

$$\theta(t) = (2\pi G/3b)^{1/2}\,\mathrm{Log}\,t. \tag{18}$$

The initial instant $t = 0$ (big bang) is pushed back to $-\infty$. We recognize here what Milne (1948) has called cosmological time.

Leaving the cosmological interpretation of our elliptic explosive-implosive or parabolic explosive systems, we shall apply the concept of internal duration and internal time to a universe of a rather different type. We consider the case of a flat space ($k = 0$) with no cosmic repulsion term ($\Lambda = 0$) and only matter ($\rho(t) = a/R^3(t)$), we have according to equation (14)

$$(\mathrm{d}R(t)/\mathrm{d}t)^2 = 8\pi G/3\,a/R(t), \quad R(0) = 0,$$

or after integration,

$$R(t) = (8\pi G/3a)^{1/3}(2/3)^{2/3}\,t^{2/3}.$$

The internal duration of interval $(t_1, t_2)$ is then given by integration of $(\mathrm{d}R(t)/\mathrm{d}t)^2$

$$\mathrm{d}(t_1, t_2) = 3(4\pi Ga)^{2/3}((t_2)^{1/3} - (t_1)^{1/3})$$

and so appears an *internal time* adapted to this universe

$$\theta(t) = 3(4\pi Ga)^{2/3}\ \ t^{1/3}. \tag{19}$$

There is no push back to $-\infty$ of the initial instant $t = 0$ and the life of this universe goes from 0 to $+\infty$ as well in terms of reference time $t$ as in terms of internal time $\theta$.

We must remark that, in all cases (cosmological or not) the push back, or non-push back, to $-\infty$ of the initial reference instant $t = 0$ depends on the behaviour of the state near the initial instant. In a similar way the push forward to $+\infty$ of the reference final instant depends on the behaviour of the state near this final instant. More precisely, considering only the first case, it is easy to see that if the state behaves like $t^{\alpha}$ near the initial instant ($t = 0$) the push back to $-\infty$ happens only if $\alpha\,\varepsilon\,]0, 1/2]$. In the cosmological models considered above push back corresponded to $\alpha = 1/2$ and non-push back to $\alpha = 2/3$.

### 5. Diffusion of heat

Vector $\mathbf{y}(t)$, representing the state of the system at reference instant $t$, does not belong necessarily to $R^{\mathrm{N}}$ (or $C^{\mathrm{N}}$) as it is the case when $\mathbf{y}$ is a solution of a differential equation. It may be a function, defined at instant $t$, of point $x$ of space $R^3$ for example. In other terms we may have, as a dynamical system, a *space-time field* $\mathbf{y}(t, x)$ satisfying a partial derivative equation. In this new case we must consider the square of the (Hermitian) norm of the partial derivative $\partial \mathrm{y}(t, x)/\partial t$ considered as a function of $x$, that is to say

$$\int_{R^3} \|\partial \mathrm{y}(t, x)/\partial t\|^2 \mathrm{d}x. \tag{20}$$

Its value, at instant $t$, is the weight of reference instant $t$. And so the *internal duration* of interval $(t_1, t_2)$ is given by

$$\mathrm{d}(t_1, t_2) = \int_{t_1,t_2} \int_{R^3} \|\partial \mathrm{y}(t, x)/\partial t\|^2 \mathrm{d}x\, \mathrm{d}t \tag{21}$$

and the *internal time* by

$$\theta(t) = \mathrm{d}(t_0, t).$$

We shall apply these definitions to a space-time field of temperatures, so to the diffusion of heat, considered, for the sake of simplicity, in the case of a space of dimension 1. Temperature at point $x$, at reference instant $t$ is $t\ u(t, x)$. If at the initial reference instant $t = 0$, the repartition of temperature is given by the $\delta$ distribution $\delta(2)$ centred at $x = 0$ (in other terms, with a rather abusive simplification, if the temperature is everywhere equal to zero except at $x = 0$ where it is infinite) the repartition of temperature at instant $t$ is classically given by

$$u(t, x) = (4\pi t)^{1/2} \exp(-x^2/4t), \tag{22}$$

which is a Laplace-Gauss function. When $t$ tends to $+\infty$, this function of $x$ flattens and tends to what we call *epsilon distribution* $\varepsilon(x)$ (Vallée, 1992). We have

$$(\partial u(t, x)/\partial t)^2 = 1/16\pi(1 + x^2/2t)^2/t^3 \exp(-x^2/2t)$$

and after some calculations

$$\int_R (\partial u(t,x)/\partial t)^2 \mathrm{d}x = 3(2\pi)^{1/2}/16\, t^{-5/2}.$$

So the *internal duration* of reference interval $(t_1, t_2)$ is

$$\mathrm{d}(t_1, t_2) = \int_{t_1,t_2} 3(2\pi)^{1/2}/16\, s^{-5/2}\, \mathrm{d}s = (2\pi)^{1/2}/8 \left(t_1^{-3/2} - t_2^{-3/2}\right)$$

and an associated *internal time* is given by

$$\theta(t) = -(2\pi)^{1/2}/8\, t^{-3/2}. \tag{23}$$

The initial reference instant $t = 0$ is pushed back to $-\infty$.

## 6. Conclusion

The results obtained about an *internal time* adapted to the evolution of a dynamical system depend on the hypotheses made. So their qualitative aspects are the most important. The anamorphosis which appears between the set of *reference instants* and the set of *internal instants* may push back to $-\infty$ the initial reference instant or push forward to $+\infty$ the final reference instant. The *reference duration* and the *internal duration* of the same interval may differ. These two qualitative results may be interpreted, as we have seen, in physiology, cosmology and field theory.

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